Diagonals of separately continuous functions of \( n \) variables with values in strongly \( \sigma \)-metrizable spaces

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Abstract. We prove the result on Baire classification of mappings \( f : X \times Y \to Z \) which are continuous with respect to the first variable and belongs to a Baire class with respect to the second one, where \( X \) is a PP-space, \( Y \) is a topological space and \( Z \) is a strongly \( \sigma \)-metrizable space with additional properties. We show that for any topological space \( X \), special equiconnected space \( Z \) and a mapping \( g : X \to Z \) of the \((n-1)\)-th Baire class there exists a strongly separately continuous mapping \( f : X^n \to Z \) with the diagonal \( g \). For wide classes of spaces \( X \) and \( Z \) we prove that diagonals of separately continuous mappings \( f : X^n \to Z \) are exactly the functions of the \((n-1)\)-th Baire class. An example of equiconnected space \( Z \) and a Baire-one mapping \( g : [0,1] \to Z \), which is not a diagonal of any separately continuous mapping \( f : [0,1]^2 \to Z \), is constructed.

Keywords: diagonal of a mapping; separately continuous mapping; Baire-one mapping; equiconnected space; strongly \( \sigma \)-metrizable space

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1. Introduction

Let \( f : X^n \to Y \) be a mapping. Then the mapping \( g : X \to Y \) defined by \( g(x) = f(x,\ldots,x) \) is called a diagonal of \( f \).

Investigations of diagonals of separately continuous functions \( f : X^n \to \mathbb{R} \) were started in classical works of R. Baire [1], H. Lebesgue [14], [15] and H. Hahn [6]. They showed that diagonals of separately continuous functions of \( n \) real variables are exactly the functions of the \((n-1)\)-th Baire class. Baire classification of separately continuous functions and their analogs is intensively studied by many mathematicians (see [17], [21], [25], [16], [2], [3], [9]).

In [16] the problem on a construction of separately continuous functions of \( n \) variables with a given diagonal of the \((n-1)\)-th Baire class was solved. It was proved in [18] that for any topological space \( X \) and a function \( g : X \to \mathbb{R} \) of the \((n-1)\)-th Baire class there exists a separately continuous function \( f : X^n \to \mathbb{R} \) with the diagonal \( g \). Further development of these investigations deals with the changing of the range space \( \mathbb{R} \) by a more general space, in particular, by a metrizable space. Notice that conditions on spaces similar to the arcwise connectedness

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(i.e., the equiconnectedness) serve as a convenient tool in a construction of separately continuous mappings (see [10, 20]).

In the given paper we study mappings \( f : X^n \to Z \) with values in a space \( Z \) from a wide class of spaces which contains metrizable equiconnected spaces and strict inductive limits of sequences of closed locally convex metrizable subspaces. We first generalize a result from [10] concerning mappings of two variables with values in a metrizable equiconnected space to the case of mappings of \( n \) variables with values in spaces from wider class. Namely, we prove a theorem on the existence of a separately continuous mapping \( f : X^n \to Z \) with the given diagonal \( g : X \to Z \) of the \((n - 1)\)th Baire class in case \( X \) is a topological space and \((Z, \lambda)\) is a strongly \(\sigma\)-metrizable equiconnected space with a perfect stratification \((Z_k)_{k=1}^{\infty}\) assigned with a mapping \(\lambda\) (Theorem 6). We also obtain a result on a Baire classification of separately continuous mappings and their analogs defined on a product of a \( PP \)-space and a topological space and with values in a strongly \(\sigma\)-metrizable space with some additional properties (Theorem 15). In order to prove this theorem we apply the technics of \(\sigma\)-discrete mappings introduced in [7] and developed in [5], [26]. For \( PP \)-spaces \( X \) using Theorem 15 we generalize Theorem 3.3 from [10] and get a characterization of diagonals of separately continuous mappings \( f : X^n \to Z \) (Theorem 16). Finally, we give an example of an equiconnected space \( Z \) and a Baire-one mapping \( g : [0, 1] \to Z \) which is not a diagonal of any separately continuous mapping \( f : [0, 1]^2 \to Z \) (Proposition 18).

2. Preliminaries

Let \( X, Y \) be topological spaces and \( C(X, Y) = B_0(X, Y) \) be the collection of all continuous mappings between \( X \) and \( Y \). For \( n \geq 1 \) we say that a mapping \( f : X \to Y \) belongs to the \( n \)th Baire class if \( f \) is a pointwise limit of a sequence \((f_k)_{k=1}^{\infty}\) of mappings \( f_k : X \to Y \) from the \((n - 1)\)th Baire class. By \( B_n(X, Y) \) we denote the collection of all mappings \( f : X \to Y \) of the \( n \)th Baire class.

For a mapping \( f : X \times Y \to Z \) and a point \((x, y) \in X \times Y\) we write \( f^y(x) = f(x, y) \). By \( CB_n(X \times Y, Z) \) we denote the collection of all mappings \( f : X \times Y \to Z \) which are continuous with respect to the first variable and belongs to the \( n \)th Baire class with respect to the second one. If \( n = 0 \), then we use the symbol \( CC(X \times Y, Z) \) for the class of all separately continuous mappings. Now let \( CC_0(X \times Y, Z) = CC(X \times Y, Z) \) and for \( n \geq 1 \) let \( CC_n(X \times Y, Z) \) be the class of all mappings \( f : X \times Y \to Z \) which are pointwise limits of a sequence of mappings from \( CC_{n-1}(X \times Y, Z) \).

For a metric space \( X \) with a metric \( | \cdot \cdot |_X \), a set \( \emptyset \neq A \subseteq X \) and a point \( x_0 \in X \) we write \( |x_0 - A|_X = \inf \{|x_0 - a|_X : a \in A\} \). If \( \delta > 0 \), then we put \( B(A, \delta) = \{x \in X : |x - A|_X < \delta\} \) and \( B[A, \delta] = \{x \in X : |x - A|_X \leq \delta\} \). If \( A = \emptyset \), then \( B(A, \delta) = B[A, \delta] = \emptyset \).

Let \( X \) be a set and \( n \in \mathbb{N} \). We denote \( \Delta_n = \{(x, \ldots, x) \in X^n : x \in X\} \). Let \( X \) be a topological space and \( \Delta = \Delta_2 = \{(x, x) : x \in X\} \). A set \( A \subseteq X \) is called equiconnected in \( X \) if there exists a continuous mapping \( \lambda : ((X \times X) \cup \Delta) \times [0, 1] \to X \) such that \( \lambda(A \times A \times [0, 1]) \subseteq A \), \( \lambda(x, y, 0) = \lambda(y, x, 1) = x \)
for all \( x, y \in A \) and \( \lambda(x, x, t) = x \) for all \( x \in X \) and \( t \in [0, 1] \). A space is equiconnected if it is equiconnected in itself. Notice that any topological vector space is equiconnected, where a mapping \( \lambda \) is defined by \( \lambda(x, y, t) = (1 - t)x + ty \).

If \( (X, \lambda) \) is an equiconnected space, then we denote \( \lambda_1 = \lambda \) and for every \( n \geq 2 \) we define a continuous function \( \lambda_n : X^{n+1} \times [0, 1]^n \to X \),

\[
(1) \quad \lambda_n(x_1, \ldots, x_{n+1}, t_1, \ldots, t_n) = \lambda(x_1, \lambda_{n-1}(x_2, \ldots, x_{n+1}, t_2, \ldots, t_n), t_1).
\]

A topological space \( X \) is called strongly \( \sigma \)-metrizable if there exists an increasing sequence \( (X_n)_{n=1}^{\infty} \) of closed metrizable subspaces \( X_n \) of \( X \) such that \( X = \bigcup_{n=1}^{\infty} X_n \) and for any convergent sequence \( (x_n)_{n=1}^{\infty} \) in \( X \) there exists a number \( m \in \mathbb{N} \) such that \( \{x_n : n \in \mathbb{N}\} \subseteq X_m \); the sequence \( (X_n)_{n=1}^{\infty} \) is called a stratification of \( X \).

We say that a family \( A = (A_i : i \in I) \) of sets \( A_i \) refines a family \( B = (B_j : j \in J) \) of sets \( B_j \) and denote it by \( A \prec B \) if for every \( i \in I \) there exists \( j \in J \) such that \( A_i \subseteq B_j \). By \( \bigcup A \) we denote the set \( \bigcup_{i \in I} A_i \).

The following notion was introduced in [23]. A space \( X \) is said to be a PP-space if there exists a sequence \( ((h_{n,i} : i \in I_n))_{n=1}^{\infty} \) of locally finite partitions of unity \( (h_{n,i} : i \in I_n) \) on \( X \) and sequence \( (\alpha_n)_{n=1}^{\infty} \) of families \( \alpha_n = (x_{n,i} : i \in I_n) \) of points \( x_{n,i} \in X \) such that for any \( x \in X \) and a neighborhood \( U \) of \( x \) there exists \( n_0 \in \mathbb{N} \) such that \( x_{n,i} \in U \) if \( n \geq n_0 \) and \( x \in \text{supp} h_{n,i} \), where \( \text{supp} h = \{x \in X : h(x) \neq 0\} \). Notice that the notion of a PP-space is close to the notion of a quarter-stratifiable space introduced in [2]. In particular, Hausdorff PP-spaces are exactly metrically quarter-stratifiable spaces [19].

Let \( A \) be a family of functionally closed subsets of a topological space \( X \). Define classes \( \mathcal{F}_\alpha \) and \( \mathcal{G}_\alpha \) as the following: \( \mathcal{F}_0 = A, \mathcal{G}_0 = \{X \setminus A : A \in A\} \) and for all \( 1 \leq \alpha < \omega_1 \) we put \( \mathcal{F}_\alpha = \{\bigcap_{n=1}^{\infty} A_n : A_n \in \bigcup_{\beta < \alpha} \mathcal{G}_\beta, n = 1, 2, \ldots\} \), \( \mathcal{G}_\alpha = \{\bigcup_{n=1}^{\infty} A_n : A_n \in \bigcup_{\beta < \alpha} \mathcal{F}_\beta, n = 1, 2, \ldots\} \). Element of families \( \mathcal{F}_\alpha \) and \( \mathcal{G}_\alpha \) are called sets of the functionally multiplicative class \( \alpha \) or sets of the functionally additive class \( \alpha \), respectively; elements of the family \( \mathcal{F}_\alpha \cap \mathcal{G}_\alpha \) are called functionally ambiguous sets of the class \( \alpha \).

A family \( A = (A_i : i \in I) \) of subsets of a topological space \( X \) is called strongly functionally discrete if there exists a discrete family \( (U_i : i \in I) \) of functionally open subsets of \( X \) such that \( \overline{A_i} \subseteq U_i \) for every \( i \in I \); \( \sigma \)-strongly functionally discrete if there exists a sequence of strongly functionally discrete families \( A_n \) such that \( A = \bigcup_{n=1}^{\infty} A_n \); a base for a mapping \( f : X \to Y \) if the preimage \( f^{-1}(V) \) of any open set \( V \) in \( Y \) is a union of sets from \( A \). By \( \Sigma_\alpha(X, Y) \) we denote the collection of all mappings between \( X \) and \( Y \) with \( \sigma \)-strongly functionally discrete bases which consist of functionally ambiguous sets of the class \( \alpha \) in \( X \).

3. A construction of functions with a given diagonal

A general construction of separately continuous mapping of two variables with a given diagonal can be found in [20]:

**Theorem 1.** Let \( X \) be a topological space, \( Z \) be a Hausdorff space, \((Z_1, \lambda)\) be an equiconnected subspace of \( Z \), \( g : X \to Z \), \((G_n)_{n=0}^{\infty} \) and \((F_n)_{n=0}^{\infty} \) be sequences
of functionally open sets $G_n$ and functionally closed sets $F_n$ in $X^2$, let $(\varphi_n)_{n=1}^{\infty}$ be a sequence of separately continuous functions $\varphi_n : X^2 \to [0,1]$, $(g_n)_{n=1}^{\infty}$ be a sequence of continuous mappings $g_n : X \to Z$ satisfying the conditions

1) $G_0 = F_0 = X^2$ and $\Delta = \{(x,x) : x \in X\} \subseteq G_{n+1} \subseteq F_n \subseteq G_n$ for every $n \in \mathbb{N}$;
2) $X^2 \setminus G_n \subseteq \varphi_n^{-1}(0)$ and $F_n \subseteq \varphi_n^{-1}(1)$ for every $n \in \mathbb{N}$;
3) $\lim_{n \to \infty} \lambda(g_n(x_n), g_{n+1}(x_n), t_n) = g(x)$ for arbitrary $x \in X$, any sequence $(x_n)_{n=1}^{\infty}$ of points $x_n \in X$ with $(x_n, x) \in F_{n-1}$ for all $n \in \mathbb{N}$, and any sequence $(t_n)_{n=1}^{\infty}$ of points $t_n \in [0,1]$. Then the mapping $f : X^2 \to Z$,

$$f(x,y) = \begin{cases} \lambda(g_n(x), g_{n+1}(x), \varphi_n(x,y)), & (x,y) \in F_{n-1} \setminus F_n \\ g(x), & (x,y) \in E = \bigcap_{n=1}^{\infty} G_n \end{cases}$$

is separately continuous.

Let $X$ be a strongly $\sigma$-metrizable space. A stratification $(X_n)_{n=1}^{\infty}$ of a space $X$ is said to be perfect if for every $n \in \mathbb{N}$ there exists a continuous mapping $\pi_n : X \to X_n$ with $\pi_n(x) = x$ for every $x \in X_n$. A stratification $(X_n)_{n=1}^{\infty}$ of an equiconnected strongly $\sigma$-metrizable space $X$ is assigned with $\lambda$ if $\lambda(X_n \times X_n \times [0,1]) \subseteq X_n$ for every $n \in \mathbb{N}$. It follows from the Dieudonné-Schwartz Theorem (see [24, Proposition II.6.5]) that a strict inductive limit of a sequence of locally convex metrizable spaces $X_n$, such that $X_n$ is closed in $X_{n+1}$, is strongly $\sigma$-metrizable space with the perfect stratification $(X_n)_{n=1}^{\infty}$ assigned with an equiconnected function $\lambda(x,y,t) = (1-t)x + ty$.

**Proposition 2.** Let $X$ be a topological space, $(Z,\lambda)$ be a strongly $\sigma$-metrizable space with a perfect stratification $(Z_n)_{n=1}^{\infty}$ assigned with a mapping $\lambda, m \in \mathbb{N}$ and $f \in B_m(X,Z)$. Then there exists a sequence $(f_n)_{n=1}^{\infty}$ of mappings $f_n \in B_{m-1}(X,Z_n)$ such that $\lim_{n \to \infty} f_n(x) = f(x)$ for every $x \in X$.

**Proof:** It is sufficient to put $f_n = \pi_n \circ g_n$, where $(\pi_n)_{n=1}^{\infty}$ is a sequence of retractions $\pi_n : Z \to Z_n$ and $(g_n)_{n=1}^{\infty}$ is a sequence of mappings $g_n \in B_{m-1}(X,Z)$ which is pointwise convergent to $f$. \qed

**Proposition 3.** Let $X$ be a metrizable space, $(Z,\lambda)$ be a strongly $\sigma$-metrizable equiconnected space with a perfect stratification $(Z_n)_{n=1}^{\infty}$ assigned with a mapping $\lambda$ and $g \in B_1(X,Z)$. Then there exists a sequence $(g_n)_{n=1}^{\infty}$ of continuous mappings $g_n : X \to Z_n$ and a sequence $(W_n)_{n=1}^{\infty}$ of open sets $W_n \subseteq X^2$ such that

1) $\Delta_2 \subseteq W_n$ for every $n \in \mathbb{N}$;
2) $\lim_{n \to \infty} g_n(x_n) = g(x)$ for every $x \in X$ and for any sequence $(x_n)_{n=1}^{\infty}$ of points $x_n \in X$ such that $(x_n, x) \in W_n$ for all $n \in \mathbb{N}$.

**Proof:** Let $(h_n)_{n=1}^{\infty}$ be a sequence of continuous mappings $h_n : X \to Z$ which is pointwise convergent to $g$ on $X$. For every $n \in \mathbb{N}$ we put $f_n = \pi_n \circ h_n$, where $(\pi_n)_{n=1}^{\infty}$ is a sequence of retractions $\pi_n : Z \to Z_n$. Clearly, $f_n \in C(X,Z_n)$.\qed
Since $Z$ is a strongly $\sigma$-metrizable space with the stratification $(Z_n)_{n=1}^\infty$, $f_n \to g$ pointwise on $X$.

For every $n \in \mathbb{N}$ we set
\[
A_n = \{ x \in X : f_k(x) \in Z_n \ \forall k \in \mathbb{N} \}.
\]
Since every $f_k$ is continuous and $Z_n$ is closed in $Z$, $A_n$ is closed in $X$ for every $n$. Moreover, $X = \bigcup_{n=1}^\infty A_n$, since $Z$ is strongly $\sigma$-metrizable.

We firstly construct a sequence $(g_n)_{n=1}^\infty$ of continuous mappings $g_n : X \to Z$ and an increasing sequence $(C_n)_{n=1}^\infty$ of closed sets $C_n \subseteq A_n$ such that $(g_n)_{n=1}^\infty$ pointwise converges to $g$ on $X$, $X = \bigcup_{n=1}^\infty C_n$ and
\[
(\forall n, k \in \mathbb{N})(\forall x \in C_k)(\exists U \in U_x)((g_n(U) \subseteq Z_k)),
\]
where by $U_x$ we denote a system of all neighborhoods of $x$ in $X$.

Let $n \in \mathbb{N}$. Define $A_0 = C_0 = \emptyset$, $F_k,n = A_k \setminus B(A_{k-1}, \frac{1}{n})$ for every $k \in \{1, \ldots, n\}$ and $C_n = \bigcup_{k=1}^n F_k,n$. Observe that every set $F_k,n$ is closed, for every $n$ the sets $F_{1,n}, \ldots, F_{n,n}$ are disjoint, every set $C_n$ is closed, $C_n \subseteq A_n \cap C_{n+1}$ for every $n$ and
\[
\bigcup_{n=1}^\infty C_n = \bigcup_{k=1}^\infty \bigcup_{n=k}^\infty F_k,n = \bigcup_{k=1}^\infty \bigcup_{n=k}^\infty A_k \setminus B(A_{k-1}, \frac{1}{n}) = \bigcup_{k=1}^\infty A_k \setminus A_{k-1} = X.
\]
For every $n \in \mathbb{N}$ we choose a family $(G_k,n : 1 \leq k \leq n)$ of open sets such that $F_k,n \subseteq G_k,n$ and sets $G_{1,n}, \ldots, G_{n,n}$ are mutually disjoint. Moreover, we take a family $(\varphi_{k,n} : 1 \leq k \leq n)$ of continuous mappings $\varphi_{k,n} : X \to [0,1]$ such that $\varphi_{k,n}(G_k,n) \subseteq \{0\}$ and $\varphi_{k,n}(G_{i,n}) \subseteq \{1\}$ for $i \neq k$. Let
\[
g_n(x) = \lambda_{n-1}(\pi_1(f_n(x)), \ldots, \pi_n(f_n(x)), \varphi_1(x), \ldots, \varphi_{n-1}(x)).
\]
Notice that every $g_n$ is continuous and $g_n \in C(X, Z_n)$ since the stratification $(Z_n)_{k=1}^\infty$ is assigned with $\lambda$. Moreover, $g_n(G_k,n) = \pi_k(f_n(G_k,n)) \subseteq Z_k$ for all $n \in \mathbb{N}$ and $k \in \{1, \ldots, n-1\}$. Since $C_k = \bigcup_{i=1}^k F_{i,k} \subseteq \bigcup_{i=1}^k F_{i,n} \subseteq \bigcup_{i=1}^k G_{i,n}$ and $g_n(\bigcup_{i=1}^k G_{i,n}) \subseteq Z_k$ for every $1 \leq k \leq n$, $(g_n)_{n=1}^\infty$ satisfies (3).

Now we show that $g_n \to g$ pointwise on $X$. Let $x_0 \in X$. Choose $k_0, n_0 \in \mathbb{N}$ such that $x_0 \in A_{k_0} \setminus A_{k_0-1}$ and $x_0 \notin B(A_{k_0-1}, \frac{1}{n_0})$. For every $n \geq \max\{k_0, n_0\}$ we have $x_0 \in F_{k_0,n}$ and $g_n(x_0) = f_n(x_0)$. In particular, $\lim_{n \to \infty} g_n(x_0) = \lim_{n \to \infty} f_n(x_0) = g(x_0)$.

By the Hausdorff Theorem on extension of metric [4, 4.5.20(c)] we choose a metric $\cdot - \cdot |_Z$ on $Z$ such that the restriction of this metric on every space $Z_n$ generates its topology. Fix $n \in \mathbb{N}$. According to (3) for every $x \in C_k \setminus C_{k-1}$ we find an open neighborhood $U_{n,x}$ of $x$ in $X$ such that
(a) $U_{n,x} \cap C_{k-1} = \emptyset$;
(b) $g_n(u) \in Z_k$ for every $u \in U_{n,x}$;
(c) $|g_n(u) - g_n(x)| < \frac{1}{n}$ for every $u \in U_{n,x}$.
Set $W_n = \bigcup_{x \in X} (U_{n,x} \times U_{n,x})$. Clearly, $(W_n)_{n=1}^{\infty}$ satisfies the condition 1). We verify 2). Let $x \in C_k \setminus C_{k-1}$ and $(x_n)_{n=1}^{\infty}$ be a sequence of points $x_n \in X$ such that $(x_n, x) \in W_n$ for every $n \in \mathbb{N}$. We choose $u_n \in X$ such that $(x_n, x) \in U_{n,u_n} \times U_{n,u_n}$, i.e. $x, x_n \in U_{n,u_n}$ for every $n \in \mathbb{N}$. It follows from (a) that $u_n \in C_k$ and the condition (b) implies that $g_n(x_n) \in Z_k$. Moreover, by (c) we have $|g_n(x_n) - g_n(x)|_Z < \frac{2}{n}$. Hence, $\lim_{n \to \infty} (|g_n(x_n) - g_n(x)|_Z = 0$. It remains to observe that the restriction of $|\cdot|_Z$ on $Z_k$ generates its topological structure. □

A schema of the proof of the following theorem was proposed by H. Hahn for functions of $n$ real variables and was applied in [16, Theorem 3.24] for mappings $f : X^n \to \mathbb{R}$.

**Theorem 4.** Let $X$ be a metrizable space, $(Z, \lambda)$ be a strongly $\sigma$-metrizable equiconnected space with a perfect stratification $(Z_k)_{k=1}^{\infty}$ assigned with $\lambda, n \in \mathbb{N}$ and $g \in B_{n-1}(X, Z)$. Then there exists a separately continuous mapping $f : X^n \to Z$ with the diagonal $g$.

**Proof:** Let $\cdot \cdot \cdot |_X$ be a metric on $X$ which generates its topological structure.

We will argue by the induction on $n$. Let $n = 2$. By Proposition 3 there exists a sequence $(g_n)_{n=1}^{\infty}$ of continuous mappings $g_n : X \to Z$ and a sequence $(W_n)_{n=1}^{\infty}$ of open sets $W_n \subseteq X^2$ which satisfy conditions 1) and 2) of Proposition 3. Now we choose sequences $(G_n)_{n=0}^{\infty}$ and $(F_n)_{n=0}^{\infty}$ of functionally open sets $G_n$ and functionally closed sets $F_n$ in $X^2$, and a sequence $(\varphi_n)_{n=1}^{\infty}$ of continuous functions $\varphi_n : X^2 \to [0, 1]$ which satisfy the first two conditions of Theorem 1 and $F_{n-1} \subseteq W_n \cap W_{n+1}$ for every $n \geq 2$. It remains to check the condition 3) of Theorem 1.

Let $x \in X$, $(x_n)_{n=1}^{\infty}$ be a sequence of points $x_n \in X$ such that $(x_n, x) \in F_{n-1}$ for every $n \in \mathbb{N}$ and $(t_n)_{n=1}^{\infty}$ be a sequence of points $t_n \in [0, 1]$. Denote $z_0 = g(x)$ and fix a neighborhood $W_0$ of $z_0$ in $Z$. Since $\lambda$ is continuous and $\lambda(z_0, z_0, t) = z_0$ for every $t \in [0, 1]$, there exists a neighborhood $W$ of $z_0$ such that $\lambda(z_1, z_2, t) \in W_0$ for any $z_1, z_2 \in W$ and $t \in [0, 1]$. By the condition 2) of Proposition 3 the equality $\lim_{n \to \infty} g_n(x_n) = \lim_{n \to \infty} g_{n+1}(x_n) = z_0$ holds. Hence, there exists $n_0 \in \mathbb{N}$ such that $g_n(x_n), g_{n+1}(x_n) \in W$ for every $n \geq n_0$. Therefore, $\lambda(g_n(x_n), g_{n+1}(x_n), t_n) \in W_0$ and $\lim_{n \to \infty} \lambda(g_n(x_n), g_{n+1}(x_n), t_n) = g(x)$. The theorem is proved for $n = 2$.

Now assume that $n \geq 3$ and suppose that the theorem is true for mappings of $(n-1)$ variables with diagonals of the $(n-2)$-th Baire class. We will prove that the theorem is true for mappings of $n$ variables with diagonals of the $(n-1)$-th Baire class.

Take a sequence $(g_k)_{k=1}^{\infty}$ of mappings $g_k \in B_{n-2}(X, Z)$ such that $g_k \to g$ pointwise on $X$. By the inductive assumption for every $k \in \mathbb{N}$ there exists a separately continuous mapping $f_k : X^{n-1} \to Z$ with the diagonal $g_k$. We put $G_0 = F_0 = X^n$, 

$$G_k = \{(x_1, \ldots, x_n) \in X^n : \max_{1 \leq i, j \leq n} |x_i - x_j|_X < \frac{1}{k}\}$$
Denote $\psi = F$ for any $t$ in every point of the set $\Delta_k$. Fix $h \subseteq G \subseteq \overline{G} \subseteq F_{k-1}$ for every $k \in \mathbb{N}$ and $\bigcap_{k=0}^{\infty} F_k = \bigcap_{k=0}^{\infty} G_k = \Delta_n$. Moreover, we choose a sequence $(\varphi_k)_{k=1}^{\infty}$ of continuous mappings $\varphi_k : X^n \to [0, 1]$ such that $X^n \setminus G_k \subseteq \varphi_k^{-1}(0)$ and $F_k \subseteq \varphi_k^{-1}(1)$ for every $k \in \mathbb{N}$.

Fix $i \in \{1, \ldots, n\}$. For any $x = (x_1, \ldots, x_n) \in X^n$ we put

$$\tilde{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Denote

$$D_i = \{ x \in X^n : \tilde{x}_i \in \Delta_{n-1} \}.$$ Notice that a function $\psi_i : X^n \setminus \Delta_n \to [0, 1]$ defined by

$$\psi_i(x_1, \ldots, x_n) = \frac{\max\{|x_j - x_k| : 1 \leq j < k \leq n, j, k \neq i\}}{\max\{|x_j - x_k| : 1 \leq j < k \leq n\}}$$

is continuous, $\psi_i(x) = 0$ if $x \in D_i \setminus \Delta_n$ and $\psi_i(x) = 1$ if $x \in D_j \setminus \Delta_n$ for $j \neq i$.

Consider a mapping $h_i : X^n \to Z$,

$$h_i(x) = \begin{cases} \lambda(f_k(\tilde{x}_i), f_{k+1}(\tilde{x}_i), \varphi_k(x)), & x \in F_{k-1} \setminus F_k, \\ g(u), & x = (u, \ldots, u) \in \Delta_n. \end{cases}$$

It is easy to see that

$$h_i(x) = \lambda(\mathcal{L}(f_k(\tilde{x}_i), f_{k+1}(\tilde{x}_i), \varphi_k(x)), f_{k+2}(\tilde{x}_i), \varphi_{k+1}(x))$$

for all $k \in \mathbb{N}$ and $x \in F_{k-1} \setminus F_{k+1}$.

Since the mappings $\lambda$, $\varphi_k$ and $\varphi_{k+1}$ are continuous and the mappings $f_k$, $f_{k+1}$ and $f_{k+2}$ are separately continuous, we get that $h_i$ is separately continuous on the open set $G_k \setminus F_{k+1}$ for every $k \in \mathbb{N}$. Moreover, $h_i$ is separately continuous on the open set $G_0 \setminus F_1 = F_0 \setminus F_1$. Then $h_i$ is separately continuous on the open set $X^n \setminus \Delta_n = \bigcup_{k=1}^{\infty} (G_k \setminus F_k)$.

We show that the mapping $h_i$ is continuous with respect to the $i$–th variable at every point of the set $\Delta_n$. Let $u \in X$, $x = (u, \ldots, u) \in \Delta_n$, $z_0 = h_i(x) = g(u)$ and $W_0$ be a neighborhood of $z_0$ in $Z$. Since $\lambda$ is continuous and $\lambda(z_0, z_0, t) = z_0$ for every $t \in [0, 1]$, there exists a neighborhood $W$ of $z_0$ such that $\lambda(z_1, z_2, t) \in W$ for any $z_1, z_2 \in W$ and $t \in [0, 1]$. Taking into consideration that $\lim_{k \to \infty} g_k(u) = g(u) = z_0$ we obtain that there exists a number $k_0$ such that $g_k(u) \in W$ for every $k \geq k_0$. Now we take any $v \in X$ such that $v \neq u$, $y = (x_1, \ldots, x_n) \in F_{k_0-1}$, where $x_j = u$ for $j \neq i$ and $x_i = v$. Choose $k \geq k_0$ with $y \in F_{k-1} \setminus F_k$. Then

$$h_i(y) = \lambda(f_k(\tilde{y}_i), f_{k+1}(\tilde{y}_i), \varphi_k(y)) = \lambda(g_k(u), g_{k+1}(u), \varphi_k(y)) \in W_0.$$
Consider a mapping \( f : X^n \to Z \),
\[
(6) \quad f(x) = \begin{cases} 
\lambda_{n-1}(h_1(x), \ldots, h_n(x), \psi_1(x), \ldots, \psi_{n-1}(x)), & x \in X^n \setminus \Delta_n \\
g(u), & x = (u, \ldots, u) \in \Delta_n.
\end{cases}
\]

Since the mappings \( h_1, \ldots, h_n \) are separately continuous and the mappings \( \lambda_{n-1}, \psi_1, \ldots, \psi_{n-1} \) are continuous, the mapping \( f \) is separately continuous on the set \( X^n \setminus \Delta_n \). It remains to prove that \( f \) is continuous with respect to each variable \( x_i \) at each point of \( \Delta_n \).

Fix \( i \in \{1, \ldots, n\} \) and take any \( x \in D_i \setminus \Delta_n \). Since \( \psi_i(x) = 0 \) and \( \psi_j(x) = 1 \) for \( j \neq i \), properties (i) and (ii) of the function \( \lambda \) and the definition (1) of the functions \( \lambda_k \) imply the equality
\[
f(x) = \lambda_{n-1}(h_1(x), \ldots, h_n(x), \psi_1(x), \ldots, \psi_{n-1}(x)) = h_i(x).
\]
Hence, \( f|_{D_i} = h_i|_{D_i} \). Therefore, the continuity of \( f \) with respect to the \( i \)-th variable at every point of \( \Delta_n \) follows from the similar property of the mapping \( h_i \). \( \square \)

**Theorem 5.** Let \( X \) be a metrizable space, \( (Z, \lambda) \) be a strongly \( \sigma \)-metrizable equiconnected space with a perfect stratification \( (Z_k)_{k=1}^{\infty} \) assigned with \( \lambda, n \in \mathbb{N} \) and \( g \in B_n(X, Z) \). Then there exists a mapping \( f \in CB_{n-1}(X \times X, Z) \cap CC_{n-1}(X \times X, Z) \) with the diagonal \( g \).

**Proof:** For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m \) we denote \( |\alpha| = \alpha_1 + \cdots + \alpha_m \).

For \( n = 1 \) the theorem is a particular case of Theorem 4.

Assume \( n \geq 2 \). Inductively for \( m = 1, \ldots, n-1 \) we choose families \( (g_\alpha : \alpha \in \mathbb{N}^m) \) of mappings \( g_\alpha \in B_{n-m}(X, Z) \) such that
\[
(7) \quad g_\alpha(x) = \lim_{k \to \infty} g_{\alpha,k}(x)
\]
for all \( x \in X, 0 \leq m \leq n-2 \) and \( \alpha \in \mathbb{N}^m \). Notice that according to [16, Lemma 3.27] these families can be chosen such that
\[
(8) \quad g_\alpha = g_\beta,
\]
if \( \alpha = (\alpha_1, \ldots, \alpha_{m-2}, \alpha_{m-1}, \alpha_m) \in \mathbb{N}^m \) and \( \beta = (\alpha_1, \ldots, \alpha_{m-2}, \alpha_m, \alpha_{m-1}) \).

For every \( \alpha \in \mathbb{N}^{n-1} \) by Proposition 3 we take sequences \( (\tilde{g}_{\alpha,k})_{k=1}^{\infty} \) of continuous mappings \( \tilde{g}_{\alpha,k} : X \to Z_k \) and \( (W_{\alpha,k})_{k=1}^{\infty} \) of open neighborhoods of the diagonal \( \Delta_2 \) which satisfy the condition 2) of Proposition 3 which we will denote by \( (2_\alpha) \). For every \( \alpha = (\alpha_1, \ldots, \alpha_{m-2}, \alpha_{m-1}, \alpha_m) \in \mathbb{N}^m \) we put \( g_\alpha = \tilde{g}_\alpha \) if \( \alpha_m \geq \alpha_{m-1} \), and \( g_\alpha = \tilde{g}_\beta \), where \( \beta = (\alpha_1, \ldots, \alpha_{m-2}, \alpha_m, \alpha_{m-1}) \) if \( \alpha_m < \alpha_{m-1} \). Notice that the family \( (g_\alpha : \alpha \in \mathbb{N}^m) \) satisfies (8), and the sequences \( (g_{\alpha,k})_{k=1}^{\infty} \) satisfy \( (2_\alpha) \). Moreover, \( g_\alpha(X) \subseteq Z_k \), where \( k = \max\{\alpha_{m-1}, \alpha_m\} \) for \( \alpha = (\alpha_1, \ldots, \alpha_{m-2}, \alpha_{m-1}, \alpha_m) \in \mathbb{N}^m \).

Let \( |\cdot|_X \) be a metric on \( X \) which generates its topological structure.
For every $\alpha \in \mathbb{N}^n$ we choose a closed neighborhood $V_\alpha \subseteq W_\alpha$ of $\Delta_2$. Put $G_0 = F_0 = X^2$. Inductively for $k \in \mathbb{N}$ we put

$$G_k = \{(x, y) \in X^2 : |x - y|/x < \frac{1}{k}\} \cap \text{int}(F_{k-1}) \cap \bigcap_{\alpha \in \mathbb{N}^n, |\alpha| \leq 2k} \{(x, y) : (y, x) \in W_\alpha\}$$

and choose a closed neighborhood $F_k$ of $\Delta$ in $X^2$ such that

$$F_k \subseteq \{(x, y) \in X^2 : |x - y|/x \leq \frac{1}{k+1}\} \cap \bigcap_{\alpha \in \mathbb{N}^n, |\alpha| \leq 2k} \{(x, y) : (y, x) \in V_\alpha\} \cap G_k.$$

Every set $G_k$ is open and

$$F_k \subseteq G_k \subseteq \overline{G_k} \subseteq F_{k-1}$$

for every $k \in \mathbb{N}$ and $\bigcap_{k=0}^\infty F_k = \bigcap_{k=0}^\infty G_k = \Delta_2$. Similarly as in the proof of Theorem 4 we choose a sequence $(\varphi_k)_{k=1}^\infty$ of continuous functions $\varphi_k : X^2 \to [0, 1]$ such that $X^2 \setminus G_k \subseteq \varphi_k^{-1}(0)$ and $F_k \subseteq \varphi_k^{-1}(1)$ for every $k \in \mathbb{N}$.

For any $m \in \{0, 1, \ldots, n-1\}$ and $\alpha \in \mathbb{N}^m$ we consider a mapping $f_\alpha : X^2 \to Z$,

$$(9) \quad f_\alpha(x, y) = \begin{cases} 
\begin{aligned}
\lambda(g_{\alpha,k}(y), g_{\alpha,k+1}(y), \varphi_k(x, y)), & (x, y) \in F_{k-1} \setminus F_k \\
g_\alpha(x), & (x, y) \in \Delta_2.
\end{aligned}
\end{cases}$$

In the same manner as in the proof of the continuity of $h_i$ with respect to the $i$–th variable in Theorem 4, by condition (7) and by the continuity of $\lambda$ and $\varphi_k$, we obtain that every $f_\alpha$ is continuous with respect to the first variable. For $\alpha \in \mathbb{N}^{n-1}$ we observe that every $f_\alpha$ is continuous with respect to the second variable on the set $X^2 \setminus \Delta_2$, since $g_{\alpha,k}$ is continuous with respect to the second variable.

Let $0 \leq m \leq n - 2$, $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ and $l \in \mathbb{N}$. It follows from (8) that

$$f_{\alpha,l}(x, y) = \begin{cases} 
\begin{aligned}
\lambda(g_{\alpha,k,l}(y), g_{\alpha,k+1,l}(y), \varphi_k(x, y)), & (x, y) \in F_{k-1} \setminus F_k \\
g_{\alpha,l}(x), & (x, y) \in \Delta_2.
\end{aligned}
\end{cases}$$

Letting $l \to \infty$, applying continuity of $\lambda$ and conditions (7), (9), we get

$$f_\alpha(x, y) = \lim_{l \to \infty} f_{\alpha,l}(x, y).$$

It remains to check that the mappings $f_\alpha$, $\alpha \in \mathbb{N}^{n-1}$, are continuous with respect to the second variable on the set $\Delta_2$. Fix $\alpha \in \mathbb{N}^{n-1}$ and $x \in X$. Let $z_0 = g_\alpha(x)$ and $W_0$ be a neighborhood of $z_0$ in $Z$. Since $\lambda(z_0, z_0, t) = z_0$ for every $t \in [0, 1]$ and the mapping $\lambda$ is continuous, there exists a neighborhood $W$ of $z_0$ such that $\lambda(z_1, z_2, t) \in W_0$ for any $z_1, z_2 \in W$ and $t \in [0, 1]$. We show that there exists $k_0 \in \mathbb{N}$ such that $\lambda(g_{\alpha,k}(y), g_{\alpha,k+1}(y), \varphi_k(x, y)) \in W_0$ for all $y \in X$ with $(x, y) \in F_{k-1} \setminus F_k$ for $k \geq k_0$. It is sufficient to prove that $g_{\alpha,k}(y), g_{\alpha,k+1}(y) \in W$ for all $y \in X$ with $(x, y) \in F_{k-1} \setminus F_k$ for $k \geq k_0$.

Assume the contrary. Then there exists a strictly increasing sequence $(k_i)_{i=1}^\infty$ of numbers $k_i$ and a sequence $(y_i)_{i=1}^\infty$ of points $y_i \in X$ such that $(x, y_i) \in F_{k_i-1} \setminus F_{k_i}$,
$g_{\alpha,k_i}(y_i) \notin W$ or $g_{\alpha,k_i+1}(y_i) \notin W$ for all $i \in \mathbb{N}$. Let $g_{\alpha,k_i}(y_i) \notin W$ for all $i \in \mathbb{N}$. We choose $i_0 \in \mathbb{N}$ such that $|\alpha,k_i| \leq 2(k_i - 1)$ for all $i \geq i_0$. Since $(x,y_i) \in F_{k_i-1}$, by the definition of $F_{k_i-1}$ it follows that $(y_i,x) \in V_{\alpha,k_i} \subseteq W_{\alpha,k_i}$. Then by condition (2) we have $\lim_{i \to \infty} g_{\alpha,k_i}(y_i) = g_\alpha(x) = z_0$, which contradicts to the condition $g_{\alpha,k_i}(y_i) \notin W$ for all $i \in \mathbb{N}$. We apply this argument again when $g_{\alpha,k_i+1}(y_i) \notin W$ for all $i \in \mathbb{N}$.

Hence, $f_\alpha$ is continuous with respect to the second variable at the point $(x,x)$, which completes the proof.

The following theorem generalizes Corollary 3.2 from [10] and Theorem 3.28 from [16].

**Theorem 6.** Let $X$ be a topological space, $(Z,\lambda)$ be a strongly $\sigma$-metrizable equiconnected space with a perfect stratification $(Z_k)_{k=1}^\infty$ assigned with $\lambda$, $n \in \mathbb{N}$ and $g \in B_n(X,Z)$. Then there exists a separately continuous mapping $f : X^{n+1} \to Z$ with the diagonal $g$ and a mapping $f_\lambda \in CB_{n-1}(X \times X,Z) \cap CC_{n-1}(X \times X,Z)$ with the diagonal $g$.

**Proof:** Let $\alpha = (\alpha_1,\ldots,\alpha_m) \in \mathbb{N}^m$ and $\alpha_{m+1} \in \mathbb{N}$. Then we will identify the multi-index $(\alpha_1,\ldots,\alpha_{m+1}) \in \mathbb{N}^{m+1}$ with the pair $\alpha,\alpha_{m+1}$. For $m = 0$ we suppose that $\mathbb{N}^0 = \{\emptyset\}$ and $h_\alpha = h$ for any mapping $h$ and $\alpha \in \mathbb{N}^0$.

Successively for $m = 1,\ldots,n$ we choose families $(g_\alpha : \alpha \in \mathbb{N}^m)$ of mappings $g_\alpha \in B_{n-m}(X,Z)$ such that

$$g_\alpha(x) = \lim_{k \to \infty} g_{\alpha,k}(x)$$

for all $x \in X$, $0 \leq i \leq n-1$ and $\alpha \in \mathbb{N}^i$. According to Proposition 2 we may assume without loss of generality that $g_{\alpha,k} \in C(X,Z_k)$ for any $\alpha \in \mathbb{N}^{n-1}$ and $k \in \mathbb{N}$.

Consider a continuous mapping

$$\varphi = \Delta_{\alpha \in \mathbb{N}^n} g_\alpha : X \to Z^{\mathbb{N}^n},$$

$\varphi(x) = (g_\alpha(x))_{\alpha \in \mathbb{N}^n}$. Denote $Y = \varphi(X)$. Since $g_\alpha(X)$ is a metrizable subspace of $Z$ for every $\alpha \in \mathbb{N}^n$, $Y$ is metrizable. For every $\alpha \in \mathbb{N}^n$ we consider a continuous mapping $h_\alpha : Y \to Z$, $h_\alpha(y) = g_\alpha(x)$, where $y = \varphi(x)$, i.e.,

$$h_\alpha(\varphi(x)) = g_\alpha(x).$$

Passing to the limit in the last equality and using (10) we obtain for $m = 1,\ldots,n$ families $(h_\alpha : \alpha \in \mathbb{N}^m)$ of mappings $h_\alpha \in B_{n-m}(Y,Z)$ such that

$$h_\alpha(y) = \lim_{k \to \infty} h_{\alpha,k}(y)$$

and

$$h_\alpha(\varphi(x)) = g_\alpha(x)$$
for all $x \in X, y \in Y$, $0 \leq i \leq n - 1$ and $\alpha \in \mathbb{N}^i$.

In particular, $h \in B_n(Y, Z)$. By Theorem 4 there exists a separately continuous mapping $\tilde{h} : Y^{n+1} \to Z$ with the diagonal $h$. Now it remains to put $f(x_1, \ldots, x_{n+1}) = \tilde{h}(\varphi(x_1), \ldots, \varphi(x_{n+1}))$.

The existence of $\tilde{f}$ can be proved similarly using Theorem 5. □

**Corollary 7.** Let $X$ be a topological space, $(Z, \lambda)$ be a metrizable equiconnected space, $n \in \mathbb{N}$ and $g \in B_{n-1}(X, Z)$. Then there exists a separately continuous mapping $f : X^n \to Z$ with the diagonal $g$ and a mapping $h \in CB_{n-1}(X \times X, Z) \cap CC_{n-1}(X \times X, Z)$ with the diagonal $g$.

4. Baire classification of $CB_n$-mappings

**Proposition 8.** Let $X, Y$ be topological spaces and $(f_i)_{i \in I}$ be at most countable family of continuous mappings $f_i : X \to Y$ such that each space $f_i(X)$ is metrizable. Then there exists a metrizable space $Z$, a continuous surjective mapping $\varphi : X \to Z$ and a family $(g_i)_{i \in I}$ of continuous mappings $g_i : Z \to Y$ such that $f_i(x) = g_i(\varphi(x))$ for all $i \in I$ and $x \in X$.

**Proof:** Consider a continuous mapping

$$\varphi = \Delta_{i \in I} f_i : X \to Y^I,$$

$\varphi(x) = (f_i(x))_{i \in I}$, and denote $Z = \varphi(X)$. Since each space $f_i(X)$ is metrizable, $Z$ is metrizable. It remains to put $g_i(z) = z_i$, where $z = (z_j)_{j \in I} \in Z$. □

**Proposition 9.** Let $X$ be a topological space and $Y$ be a metrizable space. Then

$$B_n(X, Y) \subseteq \Sigma_n^f(X, Y)$$

for every $n \in \mathbb{N}$.

**Proof:** Consider a mapping $f \in B_n(X, Y)$ and let $(f_{k_1 k_2 \ldots k_n} : k_1, k_2, \ldots, k_n \in \mathbb{N})$ be a family of continuous mappings $f_{k_1 k_2 \ldots k_n} : X \to Y$ such that

$$\lim_{k_1 \to \infty} \lim_{k_2 \to \infty} \ldots \lim_{k_n \to \infty} f_{k_1 k_2 \ldots k_n}(x) = f(x)$$

for every $x \in X$. According to Proposition 8 we choose a metrizable space $Z$, a continuous surjective mapping $\varphi : X \to Z$ and a family $(g_{k_1 k_2 \ldots k_n} : k_1, k_2, \ldots, k_n \in \mathbb{N})$ of continuous mappings $g_{k_1 k_2 \ldots k_n} : Z \to Y$ such that

$$f_{k_1 k_2 \ldots k_n}(x) = g_{k_1 k_2 \ldots k_n}(\varphi(x))$$

for all $x \in X$ and $k_1, \ldots, k_n \in \mathbb{N}$. Now for every $z = \varphi(x) \in Z$ we put

$$g(z) = \lim_{k_1 \to \infty} \lim_{k_2 \to \infty} \ldots \lim_{k_n \to \infty} g_{k_1 k_2 \ldots k_n}(z) = \lim_{k_1 \to \infty} \lim_{k_2 \to \infty} \ldots \lim_{k_n \to \infty} f_{k_1 k_2 \ldots k_n}(x) = f(x).$$
Hence, \( g \in B_n(Z, Y) \). If follows from [8] that \( g \in \Sigma_n^f(Z, Y) \). Since \( \varphi \) is continuous, \( f \in \Sigma_n^f(X, Y) \). \( \square \)

**Proposition 10.** Let \( X \) be a PP-space, \( Y \) be a topological space, \( Z \) be a metrizable space and \( n \in \mathbb{N} \cup \{0\} \). Then

\[
CB_n(X \times Y, Z) \subseteq \Sigma_{n+1}^f(X \times Y, Z).
\]

**Proof:** Let \( f \in CB_n(X \times Y, Z) \). Consider a homeomorphic embedding \( \psi : Z \to \ell_\infty \) and denote \( g = \psi \circ f \). Then \( g \in CB_n(X \times Y, \psi(Z)) \subseteq B_{n+1}(X \times Y, \ell_\infty) \) by [22, Theorem 1]. Applying Proposition 9 we obtain that \( g \in \Sigma_{n+1}^f(X \times Y, \psi(Z)) \). Since \( \psi : Z \to \psi(Z) \) is a homeomorphism, \( f \in \Sigma_{n+1}^f(X \times Y, Z) \). \( \square \)

**Proposition 11.** Let \( X \) be a topological space, \( (Y, | \cdot - \cdot |_Y) \) be a metric arcwise connected space, \( f : X \to Y \) be a mapping, \( (\mathcal{F}_k : 1 \leq k \leq n) \) be a family of strongly functionally discrete families \( \mathcal{F}_k = (F_{i,k} : i \in I_k) \) of functionally closed sets \( F_{i,k} \) in \( X \) such that \( \mathcal{F}_{k+1} \prec \mathcal{F}_k \) and for every \( i \in I_k \) and \( x_1, x_2 \in F_{i,k} \) there exists a continuous mapping \( \gamma : [0, 1] \to Y \) with \( \gamma(0) = f(x_1), \gamma(1) = f(x_2) \) and \( \text{diam}(\gamma([0, 1])) < \frac{1}{2^k} \) for every \( k \). Then there exists a continuous mapping \( g : X \to Y \) such that the inclusion \( x \in \cup \mathcal{F}_k \) for \( k = 1, \ldots, n \) implies

\[
|f(x) - g(x)|_Y < \frac{1}{2^k}.
\]

**Proof:** Take a discrete family \( (U_{i,1} : i \in I_1) \) of functionally open sets in \( X \) such that \( F_{i,1} \subseteq U_{i,1}, F_{i,1} = \varphi_{i,1}^{-1}(0) \) and \( X \setminus U_{i,1} = \varphi_{i,1}^{-1}(1) \), where \( \varphi_{i,1} : X \to [0, 1] \) is a continuous function, and put \( V_{i,1} = \varphi_{i,1}^{-1}([0, \frac{1}{2}]) \) for every \( i \in I_1 \). Then \( F_{i,1} \subseteq \overline{V_{i,1}} \subseteq U_{i,1} \). Now choose a discrete family \( (G_{i,2} : i \in I_2) \) of functionally open sets such that \( F_{i,2} \subseteq G_{i,2} \) for every \( i \in I_2 \). Since \( \mathcal{F}_2 \prec \mathcal{F}_1 \), for every \( i \in I_2 \) we fix a unique \( j \in I_1 \) such that \( F_{i,2} \subseteq F_{j,1} \). Let \( U_{i,2} = G_{i,2} \cap V_{j,1} \). Then \( F_{i,2} = \varphi_{i,2}^{-1}(0) \) and \( X \setminus U_{i,1} = \varphi_{i,2}^{-1}(1) \) for some continuous function \( \varphi_{i,2} : X \to [0, 1] \). Denote \( V_{i,2} = \varphi_{i,2}^{-1}([0, \frac{1}{2}]) \). Then \( F_{i,2} \subseteq \overline{V_{i,2}} \subseteq U_{i,2} \subseteq V_{j,1} \). Proceeding analogously we get discrete families \( (U_{i,k} : i \in I_k) \) and \( (V_{i,k} : i \in I_k) \) of functionally open subsets of \( X \) for \( k = 1, \ldots, n - 1 \) and \( i \in I_{k+1} \) there is a unique \( j = j_k(i) \in I_k \) with

\[
F_{i,k+1} \subseteq \overline{V_{i,k+1}} \subseteq U_{i,k+1} \subseteq V_{j,k}.
\]

For every \( k \) we put

\[
U_k = \bigcup_{i \in I_k} U_{i,k}
\]

and observe that the sets

\[
H_k = \bigcup_{i \in I_k} \varphi_{i,k}^{-1}([0, \frac{1}{2}]) \quad \text{and} \quad E_k = X \setminus U_k
\]

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are disjoint and functionally closed in $X$. Take a continuous function $h_k : X \to [0,1]$ such that $H_k = h_k^{-1}(1)$ and $E_k = h_k^{-1}(0)$.

Fix arbitrary points $y_0 \in f(X)$ and $y_{i,k} \in f(F_{i,k})$ for every $k$ and $i \in I_k$, and for all $x \in X$ put $g_0(x) = y_0$. Since $Y$ is arcwise connected, for every $i \in I_1$ there exists a continuous function $\gamma_{i,1} : [0,1] \to Y$ such that $\gamma_{i,1}(0) = y_0$ and $\gamma_{i,1}(1) = y_{i,1}$. Now for every $1 < k \leq n$ and $i \in I_k$ there exists a continuous function $\gamma_{i,k} : [0,1] \to Y$ such that $\gamma_{i,k}(0) = y_{j,k-1}$, where $j \in I_{k-1}$ satisfies $F_{i,k} \subseteq F_{j,k-1}$, $\gamma_{i,k}(1) = y_{i,k}$ and

\begin{equation}
(16) \quad \text{diam}(\gamma_{i,k}([0,1])) < \frac{1}{2^{k+1}}.
\end{equation}

Inductively for $k = 0, \ldots, n-1$ we define a continuous mapping $g_{k+1} : X \to Y$,

\[ g_{k+1}(x) = \begin{cases} 
 g_k(x), & x \in E_{k+1}, \\
 \gamma_{i,k+1}(h_{k+1}(x)), & i \in I_{k+1}, x \in U_{i,k+1}.
\end{cases} \]

Notice that $g_{k+1}(x) = y_{i,k+1}$ for all $x \in \overline{V_{i,k+1}}$ and $i \in I_{k+1}$.

We show that for all $x \in X$ the inequality

\begin{equation}
(17) \quad |g_{k+1}(x) - g_k(x)|_Y < \frac{1}{2^{k+2}}
\end{equation}

holds for $k \geq 1$. Clearly, (17) is valid if $x \in E_{k+1}$. Let $x \in U_{i,k+1}$ for $i \in I_{k+1}$. Then $g_{k+1}(x) = \gamma_{i,k+1}(h_{k+1}(x))$ and $g_k(x) = y_{j,k} = \gamma_{i,k+1}(0)$, since $x \in V_{j,k}$ for $j = j_k(i) \in I_k$. Taking into account (16) we obtain (17).

We put $g = g_n$. Let $1 \leq k \leq n$ and $x \in \bigcup F_k$. Then $x \in F_{i,k}$ for some $i \in I_k$. It follows that $g_k(x) = y_{i,k} \in f(F_{i,k})$. Then $|f(x) - g_k(x)|_Y \leq \frac{1}{2^{k+1}}$. The inequality (17) implies that

\[ |f(x) - g(x)|_Y \leq |f(x) - g_k(x)|_Y + \sum_{i=k}^{n-1} |g_i(x) - g_{i+1}(x)|_Y < \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} = \frac{1}{2^k}. \]

The similar result to the following theorem was obtained also in [13, Theorem 4.1], but we include its proof for the sake of completeness.

**Theorem 12.** Let $X$ be a topological space, $Y$ be a metrizable arcwise connected and locally arcwise connected space. Then $\Sigma_f^1(X,Y) \subseteq B_1(X,Y)$.

**Proof:** Fix a metric $|\cdot - \cdot|_Y$ on $Y$ which generates its topological structure. For every $k \in \mathbb{N}$ and $y \in Y$ we take an open neighborhood $U_k(y)$ of $y$ such that any points from $U_k(y)$ can be joined with an arc of a diameter $< \frac{1}{2^{k+1}}$.

Let $f \in \Sigma_f^1(X,Y)$. It is easy to see that $f$ has a $\sigma$-strongly functionally discrete base $\mathcal{B}$ which consists of functionally closed sets in $X$. For every $k \in \mathbb{N}$ we put

\[ \mathcal{B}_k = \{ B \in \mathcal{B} : \exists y \in Y \mid B \subseteq f^{-1}(U_k(y)) \}. \]
Then $\mathcal{B}_k$ is a $\sigma$-strongly functionally discrete family and $X = \bigcup \mathcal{B}_k$ for every $k$. According to [12, Lemma 13] for every $k \in \mathbb{N}$ there exists a sequence $(\mathcal{B}_{k,n})_{n=1}^{\infty}$ of strongly functionally discrete families $\mathcal{B}_{k,n} = (B_{k,n,i} : i \in I_{k,n})$ of functionally closed subsets of $X$ such that $\mathcal{B}_{k,n} \prec \mathcal{B}_k$ and $\mathcal{B}_{k,n} \prec \mathcal{B}_{k,n+1}$ for every $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} \bigcup \mathcal{B}_{k,n} = X$. For all $k, n \in \mathbb{N}$ we put

$$\mathcal{F}_{k,n} = (B_{1,n,i_1} \cap \cdots \cap B_{k,n,i_k} : i_m \in I_{m,n}, 1 \leq m \leq k).$$

Notice that every family $\mathcal{F}_{k,n}$ is strongly functionally discrete, consists of functionally closed sets and

(a) $\mathcal{F}_{k+1,n} \prec \mathcal{F}_{k,n},$

(b) $\mathcal{F}_{k,n} \prec \mathcal{F}_{k,n+1},$

(c) $\bigcup_{n=1}^{\infty} \bigcup \mathcal{F}_{k,n} = X.$

For every $n \in \mathbb{N}$ we apply Proposition 11 to the function $f$ and the families $\mathcal{F}_{1,n}, \mathcal{F}_{2,n}, \ldots, \mathcal{F}_{n,n}$. We obtain a sequence of continuous mappings $g_n : X \rightarrow Y$ such that

$$|f(x) - g_n(x)|_Y < \frac{1}{2^k}$$

if $x \in \mathcal{F}_{k,n}$ for $k \leq n$.

Now conditions (b) and (c) imply that $g_n \rightarrow f$ pointwise on $X$. Hence, $f \in B_1(X,Y)$. □

Let $Z$ be a topological space and $(Z_k)_{k=1}^{\infty}$ be a sequence of sets $Z_k \subseteq Z$ such that $Z = \bigcup_{k=1}^{\infty} Z_k$. We say that the pair $(Z,(Z_k)_{k=1}^{\infty})$ has the property $(\ast)$ if for every convergent sequence $(x_m)_{m=1}^{\infty}$ in $Z$ there exists a number $k$ such that $\{x_m : m \in \mathbb{N}\} \subseteq Z_k$.

**Proposition 13.** Let $X$ be a PP-space, $Y$ be a topological space, $n \in \mathbb{N} \cup \{0\}$, $(Z,(Z_k)_{k=1}^{\infty})$ have the property $(\ast)$, $Z_k$ be functionally closed in $Z$ and $f \in CB_n(X \times Y, Z)$. Then there exists a sequence $(B_k)_{k=1}^{\infty}$ of sets of the functionally multiplicative class $n$ in $X \times Y$ such that $\bigcup_{k=1}^{\infty} B_k = X \times Y$ and $f(B_k) \subseteq Z_k$ for every $k \in \mathbb{N}$.

**Proof:** Take a sequence $(U_m = (U_{i,m} : i \in I_m))_{m=1}^{\infty}$ of families of points from $X$ such that

$$(\forall x \in X)((\forall m \in \mathbb{N} \ x \in U_{i,m,m}) \implies (x_{i,m} \rightarrow x)).$$

By [19, Corollary 3.1] there exists a weaker metrizable topology $T$ on $X$ in which every $U_{i,m}$ is open. Since $(X,T)$ is paracompact, for every $m$ there exists a locally finite open covering $V_m = (V_{s,m} : s \in S_m)$ which refines $U_m$. It follows from [4, Theorem 1.5.18] that for every $m$ there exists a locally finite closed covering $(F_{s,m} : s \in S_m)$ of $(X,T)$ such that $F_{s,m} \subseteq V_{s,m}$ for every $s \in S_m$. Now for every $s \in S_m$ we choose $i_m(s) \in I_m$ such that $F_{s,m} \subseteq U_{i_m(s),m}$.
For all \( m, k \in \mathbb{N} \) and \( s \in S_m \) we denote \( i = i_m(s) \) and put

\[
A_{s,m,k} = (f^{x_i,m})^{-1}(Z_k), \quad B_{m,k} = \bigcup_{s \in S_m} (F_{s,m} \times A_{s,m,k}), \quad B_k = \bigcap_{m=1}^{\infty} B_{m,k}.
\]

Since \( f \) belongs to the \( n \)-th Baire class with respect to the second variable, for every \( k \) the set \( A_{s,m,k} \) is of the functionally multiplicative class \( n \) in \( Y \) for all \( m \in \mathbb{N} \) and \( s \in S_m \). Then the set \( B_{m,k} \) is of the functionally multiplicative class \( n \) in \( (X, T) \times Y \) as a locally finite union of sets of the \( n \)-th functionally multiplicative class. Hence, \( B_k \) is of the \( n \)-th functionally multiplicative class in \( (X, T) \times Y \), and, consequently, in \( X \times Y \) for every \( k \).

We show that \( f(B_k) \subseteq Z_k \) for every \( k \). Fix \( k \in \mathbb{N} \) and \( (x, y) \in B_k \). Take a sequence \( (s_m)_{m=1}^{\infty} \) of indexes \( s_m \in S_m \) such that \( x \in F_{s_m,m} \subseteq U_{i_m(s_m),m} \) and \( f(x_{i_m(s_m),m}, y) \in Z_k \). Then \( x_{i_m(s_m),m} \rightarrow_{m \rightarrow \infty} x \). Since \( f \) is continuous with respect to the first variable, \( f(x_{i_m(s_m),m}, y) \rightarrow_{m \rightarrow \infty} f(x, y) \). Since \( Z_k \) is closed, \( f(x, y) \in Z_k \).

It remains to show that \( \bigcup_{k=1}^{\infty} B_k = X \times Y \). Let \( (x, y) \in X \times Y \). Then there exists a sequence \( (s_m)_{m=1}^{\infty} \) of indexes \( s_m \in S_m \) such that \( s_m \in S_m \) and \( x \in F_{s_m,m} \subseteq U_{i_m(s_m),m} \). Notice that \( f(x_{i_m(s_m),m}, y) \rightarrow_{m \rightarrow \infty} f(x, y) \). Since \( (Z_k)_{k=1}^{\infty} \) satisfies \((*)\), there exists a number \( k \) such that the set \( \{ f(x_{i_m(s_m),m}, y) : m \in \mathbb{N} \} \) is contained in \( Z_k \), i.e. \( y \in A_{s_m,m,k} \) for every \( m \in \mathbb{N} \). Hence, \( (x, y) \in B_k \).

The following result will be useful (see [11, Proposition 5.2]).

**Proposition 14.** Let \( 0 < \alpha < \omega_1 \), \( X \) be a topological space, \( Z = \bigcup_{k=1}^{\infty} Z_k \) be a contractible space, \( f : X \rightarrow Z \) be a mapping, \( (X_k)_{k=1}^{\infty} \) be a sequence of sets of the \( \alpha \)-th functionally additive class in \( X \) such that \( X = \bigcup_{k=1}^{\infty} X_k \), \( f(X_k) \subseteq Z_k \) and assume that there exists a function \( f_k \in B_\alpha(X, Z_k) \) with \( f_k|_{X_k} = f|_{X_k} \) for every \( k \in \mathbb{N} \). Then \( f \in B_\alpha(X, Z) \).

**Theorem 15.** Let \( n \in \mathbb{N} \), \( X \) be a PP-space, \( Y \) be a topological space and \( Z \) be a contractible space. Then

\[
CB_n(X \times Y, Z) \subseteq B_{n+1}(X \times Y, Z).
\]

If, moreover, \( Z \) is a strongly \( \sigma \)-metrizable space with a perfect stratification \( (Z_k)_{k=1}^{\infty} \), where every \( Z_k \) is an arcwise connected and locally arcwise connected subspace of \( Z \), then

\[
CC(X \times Y, Z) \subseteq B_1(X \times Y, Z).
\]

**Proof:** By the definition of a PP-space we choose a sequence \( ((h_{n,i} : i \in I_n))_{n=1}^{\infty} \) of locally finite partitions of unity \( (h_{n,i} : i \in I_n) \) on \( X \) and a sequence \( (\alpha_n)_{n=1}^{\infty} \) of families \( \alpha_n = (x_{n,i} : i \in I_n) \) of points \( x_{n,i} \in X \) such that for any \( x \in X \) the condition \( x \in \text{supp} h_{n,i} \) implies that \( x_{n,i} \rightarrow x \). According to [19, Proposition 3.2] there exists a continuous pseudo-metric \( p \) on \( X \) such that each function \( h_{n,i} \) is continuous with respect to \( p \). Then the first inclusion \( CB_n(X \times Y, Z) \subseteq B_{n+1}(X \times Y, Z) \) holds true.
Y, Z) in fact was proved in [2, Theorem 5.3], where X is a metrically quarter-
stratifiable space (i.e., Hausdorff PP-space [19]). Another proof of this inclusion
 can be obtained analogously to the proof of Theorem 6.6 from [9].

Now we prove the second inclusion. Let $f \in CC(X \times Y, Z)$. For every $k \in \mathbb{N}$ we
consider a retraction $\pi_k : Z \to Z_k$. Notice that every subspace $Z_k$ is functionally
closed in $Z$ as the preimage of closed set under a continuous mapping $\varphi : Z \to
\prod_{k=1}^{\infty} Z_k$, $\varphi(z) = (\pi_k(z))_{k=1}^{\infty}$. By Proposition 13 we take a sequence $(B_k)_{k=1}^{\infty}$ of
functionally closed subsets of $X \times Y$ such that $\bigcup_{k=1}^{\infty} B_k = X \times Y$ and $f(B_k) \subseteq Z_k$
for every $k \in \mathbb{N}$. Observe that

$$f_k = \pi_k \circ f \in CC(X \times Y, Z_k) \subseteq \Sigma^f_1(X \times Y, Z_k)$$

by Proposition 10. According to Theorem 12, $f_k \in B_1(X \times Y, Z_k)$. Moreover,
$f_k|_{B_k} = f|_{B_k}$. It remains to notice that every set $B_k$ belongs to the first function-
ally additive class in $X \times Y$ and to apply Proposition 14.

The following result generalizes Theorem 3.3 from [10] and gives a characteriza-
tion of diagonals of separately continuous mappings.

**Theorem 16.** Let $X$ be a topological space, $(Z, \lambda)$ be a strongly $\sigma$-metrizable
equiconnected space with a perfect stratification $(Z_k)_{k=1}^{\infty}$ assigned with $\lambda$, $n \in \mathbb{N}$,
g : $X \to Z$ and at least one of the following conditions holds:

1. every separately continuous mapping $h : X^{n+1} \to Z$ belongs to the $n$–th
   Baire class;
2. $X$ is a PP-space (in particular, $X$ is a metrizable space).

Then the following conditions are equivalent:

(i) $g \in B_n(X, Z)$;
(ii) there exists a separately continuous mapping $f : X^{n+1} \to Z$ with the
diagonal $g$.

**Proof:** In the case (1) the theorem is a corollary from Theorem 6.

In the case (2) the theorem follows from Theorem 15 and case (1).

The following characterizations of diagonals of separately continuous mappings
can be proved similarly.

**Theorem 17.** Let $X$ be a topological space, $(Z, \lambda)$ be a strongly $\sigma$-metrizable
equiconnected space with a perfect stratification $(Z_k)_{k=1}^{\infty}$ assigned with $\lambda$, $n \in \mathbb{N}$,
g : $X \to Z$ and at least one of the following conditions holds:

1. every separately continuous mapping $h : X^2 \to Z$ belongs to the first
   Baire class;
2. $X$ is a PP-space (in particular, $X$ is a metrizable space).

Then the following conditions are equivalent:

(i) $g \in B_n(X, Z)$;
(ii) there exists a mapping $f \in CB_{n-1}(X \times X, Z) \cap CC_{n-1}(X \times X, Z)$ with
the diagonal $g$. 
5. Examples and questions

For a topological space $Y$ by $\mathcal{F}(Y)$ we denote the space of all nonempty closed subsets of $Y$ with the Vietoris topology.

A multi-valued mapping $f : X \to \mathcal{F}(Y)$ is said to be upper (lower) continuous at $x_0 \in X$ if for any open set $V \subseteq Y$ with $f(x_0) \subseteq V$ ($f(x_0) \cap V \neq \emptyset$) there exists a neighborhood $U$ of $x_0$ in $X$ such that $f(x) \subseteq V$ ($f(x) \cap V \neq \emptyset$) for every $x \in U$. If a multi-valued mapping $f$ is upper and lower continuous at $x_0$ simultaneously, then it is called continuous at $x_0$.

**Proposition 18.** There exists an equiconnected space $(Z, \lambda)$ with a metrizable equiconnected subspace $Z_1$ and a mapping $g \in B_1([0,1], Z)$ such that

1. there exists a sequence $(g_n)_{n=1}^{\infty}$ of continuous mappings $g_n : [0,1] \to Z_1$ which is pointwise convergent to $g$;
2. $g$ is not a diagonal of any separately continuous mapping $f : [0,1]^2 \to Z$.

**Proof:** Let $Y = [0, 1] \times [0, 1)$ and

$$Z = \{ \{x\} \times [0,y) : x \in [0,1], y \in [0,1) \} \cup \{ \{x\} \times [0,1) : x \in [0,1] \}$$

be a subspace of $\mathcal{F}(Y)$. Notice that $Z_1 = \{ \{x\} \times [0,y) : x \in [0,1], y \in [0,1) \}$ is dense metrizable subspace of $Z$, since $Z_1$ consists of compacts subsets of a metrizable space $Y$.

We show that $Z$ is equiconnected. Firstly we consider the space $Q = [0,1]^2$. For $q_1 = (x_1, y_1), q_2 = (x_2, y_2) \in Q$ we set

$$\theta(q_1, q_2) = \min \{y_1, y_2, 1 - |x_1 - x_2| \},$$

$$|\alpha_1(q_1, q_2)| = y_1 - \theta(q_1, q_2), \quad |\alpha_2(q_1, q_2)| = |x_1 - x_2|, \quad |\alpha_3(q_1, q_2)| = y_2 - \theta(q_1, q_2)$$
and $|\alpha(q_1, q_2)| = \alpha_1(q_1, q_2) + \alpha_2(q_1, q_2) + \alpha_3(q_1, q_2)$. We denote $\theta = \theta(q_1, q_2)$, $|\alpha_1(q_1, q_2)|, |\alpha_2(q_1, q_2)|, |\alpha_3(q_1, q_2)|, |\alpha(q_1, q_2)|$ and set

$$\mu(q_1, q_2, t) = \begin{cases} (x_1, y_1 - t\alpha), & q_1 \neq q_2, t \in [0, \frac{\alpha_1}{\alpha}]; \\ (x_1 + (t\alpha - \alpha_1)\text{sign}(x_2 - x_1), \theta), & q_1 \neq q_2, t \in [\frac{\alpha_1}{\alpha}, \frac{\alpha_1 + \alpha_2}{\alpha}]; \\ (x_2, \theta + t\alpha - \alpha_1 - \alpha_2), & q_1 \neq q_2, t \in [\frac{\alpha_1 + \alpha_2}{\alpha}, \frac{\alpha_1 + \alpha_2 + \alpha_3}{\alpha}]; \\ q_1, & q_1 = q_2, t \in [0, 1]. \end{cases}$$

The function $\mu : Q^2 \times [0, 1] \to Q$ is continuous and the space $(Q, \mu)$ is equiconnected.

Consider the continuous bijection $\varphi : Z \to Q$,

$$\varphi(z) = \begin{cases} (x, y), & z = x \times [0, y]; \\ (x, 1), & z = x \times [0, 1]. \end{cases}$$
Let \( \lambda \) be a continuous mapping such that \( \lambda(z_1, z_2) \in Z_1 \) and \( \lambda(z_1, z_2, t) \in Z \) for every \( z, t \in [0, 1] \). It is easy to see that \( \lim_{t \to 0} \lambda(z_1, z_2, t) = \lambda(z_1, z_2) \). Note that \( \lambda \) is upper continuous at a point \( (z_1, z_2, t) \) if \( \lambda(z_1, z_2, t) = \lambda(z_1, z_2, t_0) = \lambda(Z_1, z_2) \). Suppose that \( \lambda(z_1, z_2, t_0) = \lambda(z_1, z_2) = z_1 = x_1 \times [0, 1) \) and \( z_2 \subseteq x_2 \times [0, 1) \). Fix a set \( G \) open in \( Y \) such that \( z_1 \subseteq G \).

Let \( x_1 \neq x_2 \). Note that \( t_0 = 0 \). Choose a neighborhood \( U_1 \) of \( z_1 \), a neighborhood \( U_2 \) of \( z_2 \) and \( \delta > 0 \) such that \( z \subseteq G \) for every \( z \in U_1 \) and

\[
\frac{\alpha_1(\varphi(z'), \varphi(z''))}{\alpha(\varphi(z'), \varphi(z''))} \geq \delta
\]

for every \( z' \in U_1 \) and \( z'' \in U_2 \). According to (19), \( \lambda(z', z'', t) \subseteq G \) for every \( z' \subseteq U_1, z'' \subseteq U_2 \) and \( t \in [0, \delta) \).

Now let \( x_1 = x_2 \). Choose a set \( G_0 \) open in \( Y \) such that \( z_1 \subseteq G_0 \subseteq G \) and if \( (x', y), (x'', y) \in G_0 \), then \( \{(x') \times [0, y] \} \cup \{(x'', y) \times \{y\} \} \subseteq G_0 \). It follows from (19) that \( \lambda(z', z'', t) \subseteq G_0 \) for every \( z', z'' \subseteq G_0 \) and \( t \in [0, 1] \).

In the case of \( \lambda(z_1, z_2, t_0) = z_2 = x_2 \times [0, 1) \) we argue analogously. Thus the mapping \( \lambda \) is continuous and, consequently, \( (Z, \lambda) \) is equiconnected. Moreover, \( \lambda(Z_1 \times Z_1 \times [0, 1]) \subseteq Z_1 \). Hence, \( Z_1 \) is an equiconnected subspace of \( Z \).

We define a mapping \( g : [0, 1] \to Z \),

\[
g(x) = \{x\} \times [0, 1)
\]

and for every \( n \in \mathbb{N} \) we consider a continuous mapping \( g_n : [0, 1] \to Z_1 \),

\[
g_n(x) = \{x\} \times \left[0, 1 - \frac{1}{n}\right].
\]

It is easy to see that \( \lim_{n \to \infty} g_n(x) = g(x) \) for every \( x \in [0, 1] \), i.e. the condition (1) of the proposition holds.

Now we verify (2). Assume to the contrary that there exists a separately continuous mapping \( f : [0, 1]^2 \to Z \) such that \( f(x, x) = g(x) \) for every \( x \in X \). Since \( f \) is separately upper continuous on the set \( \Delta = \{(x, x) : x \in [0, 1]\} \), for every \( x \in [0, 1] \) there exists \( \delta_x \in (0, 1) \) such that

\[
(f(x, y) \cup f(y, x)) \cap ([0, 1] \times [1 - \delta_x, 1]) \subseteq g(x)
\]

for every \( y \in [0, 1] \) with \( |x - y| < \delta_x \).

Take \( \delta > 0 \), an open nonempty set \( U \subseteq [0, 1] \) and a set \( A \) dense in \( U \) such that \( \delta_x \geq \delta \) for every \( x \in A \). Without loss of generality we may suppose that
diam$(U) < \delta$. Then
\[ f(x, y) \cap ([0, 1] \times [1 - \delta, 1)) \subseteq g(x) \cap g(y) \]
for any $x, y \in A$. Since $g(x) \cap g(y) = \emptyset$ for any distinct $x, y \in [0, 1]$, $f(x, y) \subseteq [0, 1] \times [0, 1 - \delta]$ for any distinct $x, y \in A$. Since $f$ is separately lower continuous and $A$ is dense in $U$, $f(x, y) \subseteq [0, 1] \times [0, 1 - \delta]$ for any $x, y \in U$, which leads to a contradiction, provided $g$ is a diagonal of $f$.

**Question 1.** Let $Z$ be a topological vector space and $g \in B_1([0, 1], Z)$. Does there exist a separately continuous mapping $f : [0, 1]^2 \to Z$ with the diagonal $g$?

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